

# HALF-ISOMORPHISMS OF FINITE AUTOMORPHIC MOUFANG LOOPS.

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ABSTRACT. We show that each half-automorphism of a finite automorphic Moufang loop is trivial. In general this is not true for finite left automorphic Moufang loops and for finite automorphic loops.

## 1. INTRODUCTION.

A half-isomorphism of a loop is bijection  $\tau$  such that  $\tau(xy)$  equals either  $\tau(x)\tau(y)$  or  $\tau(y)\tau(x)$  for all elements  $x, y$ . For groups all half-isomorphisms are either an isomorphism or an anti-isomorphism as was shown by Scott [18]. Following the terminology of [18] we will call such half-isomorphisms trivial. In general for loops the situation is different: there exist Moufang loops with half-isomorphisms which are neither an isomorphism nor an anti-isomorphism. In many cases, in particular, for finite Moufang loops of odd order, all half-isomorphisms are trivial ([5],[15]). Obviously the half-isomorphisms of a loop always form a group, which is a new and, at the time of the writing this note, rather mysterious invariant of a loop. Let us note that any anti-isomorphism of a Moufang loop  $M$  to a Moufang loop  $M_1$  is the composition of an isomorphism of  $M$  to  $M_1$  and inverse map on  $M_1$ .

## 2. PRELIMINARIES AND DEFINITIONS.

In this paper we consider a *loop* as a universal algebra  $\langle Q; \cdot, \backslash, /, 1 \rangle$  of type  $(3, 0, 1)$  such that the identities

$$(x \cdot y)/y = x = (x/y) \cdot y,$$

$$x \cdot (x \backslash y) = y = x \backslash (x \cdot y),$$

$$x \cdot 1 = x = 1 \cdot x$$

hold for all  $x, y \in L$ . In the following we often write  $xy$  instead of  $x \cdot y$ .

A *Moufang loop* is a loop in which one of the following equivalent identities holds:

$$\begin{aligned} ((xy)x)z &= x(y(xz)), \\ ((xy)z)y &= x(y(zy)), \\ (xy)(zx) &= (x(yz))x. \end{aligned}$$

The bijection  $L_a : Q \rightarrow Q$ ,  $L_ax = ax$  is called a *left multiplication*, analogously the bijection  $R_a : Q \rightarrow Q$ ,  $R_ay = ya$  is called a *right multiplication* for all  $a \in Q$ .

With an arbitrary loop  $Q$  one may associate several groups of transformations of the set  $Q$ , such as:

- the multiplication group  $\text{Mlt}(Q)$  generated by the left and right multiplications by elements of  $Q$ ;
- the left multiplication group  $\text{LMlt}(Q) \subseteq \text{Mlt}(Q)$  generated by the left multiplications only;
- the inner mapping group  $\text{Inn}(Q) \subseteq \text{Mlt}(Q)$  defined as the stabilizer of the neutral element  $1 \in Q$ ;
- the left inner mapping group  $\text{LInn}(Q)$  which is the stabilizer of 1 in  $\text{LMlt}(Q)$ .

$\text{Inn}(Q)$ , the *inner mapping group* of  $Q$ , is generated by three families of elements of  $\text{Mlt}(Q)$ :

$$\begin{aligned} \ell_{x,y} &= L_{xy}^{-1} \circ L_x \circ L_y, \\ r_{x,y} &= R_{xy}^{-1} \circ R_y \circ R_x, \\ T_x &= L_x^{-1} \circ R_x \end{aligned}$$

for all  $x, y \in Q$ . The loop  $Q$  is called *automorphic* if  $\text{Inn}(Q)$  acts on  $Q$  by automorphisms, and *left automorphic* if  $\text{LInn}(Q)$  does.

A loop  $Q$  is automorphic if the mappings  $\ell_{x,y}$ ,  $r_{x,y}$  and  $T_x$  for all  $x, y \in Q$  are automorphisms of  $Q$ . A loop  $Q$  is left automorphic if the mappings  $\ell_{x,y}$  are automorphisms of  $Q$ . It is known for a Moufang loop  $Q$  that if  $\ell_{x,y}$  is an automorphism for some  $x, y \in Q$  then  $r_{x,y}$  is also an automorphism of  $Q$  and vice versa (see [1]).

The property of a loop to be left automorphic is of significance in differential geometry. In particular, a reductive homogeneous space is completely determined by a loop with the property of left power-alternativity (or left monoalternativity) and the left automorphic property ([11], [17]).

Each alternative (in particular, each diassociative) automorphic loop is Moufang ([12]). Also, each power-alternative left automorphic smooth loop is Moufang ([2]). Besides every smooth automorphic Moufang

loop is a group. A well known class of non-trivial finite examples of left automorphic Moufang loops is provided by Code loops ([7]).

As usual one defines a *normal subloop* as the kernel of a loop homomorphism. A subloop of a given loop  $Q$  is normal if and only if it is invariant under the  $\text{Inn}(Q)$ .

Moufang loops are diassociative, that is the subloop generated by two elements is a group. Hence in a Moufang loop one has

$$x \backslash y = x^{-1}y, \quad y/x = yx^{-1}.$$

In a Moufang loop  $Q$  the *commutator subloop*  $[Q, Q]$  is the subloop generated by all elements of the form  $x^{-1}y^{-1}xy = [x, y]$ , where  $x, y \in Q$  and the *associator subloop*  $(Q, Q, Q)$  is the subloop generated by all elements of the form  $(x(yz))^{-1}((xy)z) = (x, y, z)$  for all  $x, y, z \in Q$ .

The subloop  $Z(Q) = \{x \in Q : [x, a] = 1, \forall a \in Q\}$  is called the *commutant* of  $Q$ .

The subloops

$$N_\lambda(Q) = \{x \in Q : (x, a, b) = 1, \forall a, b \in Q\},$$

$$N_\mu(Q) = \{x \in Q : (a, x, b) = 1, \forall a, b \in Q\},$$

$$N_\rho(Q) = \{x \in Q : (a, b, x) = 1, \forall a, b \in Q\}$$

are called *left*, *middle* and *right nucleus* of  $Q$ . The subloop  $N(Q) = N_\lambda(Q) \cap N_\mu(Q) \cap N_\rho(Q)$  is called the *nucleus* of  $Q$ . In a Moufang loop  $L$  all nuclei coincide. The subloop  $C(Q) = Z(Q) \cap N(Q)$  is called the *center* of the loop  $Q$ .

All characteristic subloops of any automorphic loop are normal. In particular,  $[Q, Q]$ ,  $(Q, Q, Q)$ ,  $Z(Q)$ ,  $N(Q)$ ,  $C(Q)$  are normal subloops of an automorphic Moufang loop  $Q$ .

### 3. MAIN THEOREM

The aim of this note is to show the following

**Theorem 3.1.** *Let  $L$  be a finite automorphic Moufang loop and let  $\tau$  be a half-automorphism of  $L$ . Then  $\tau$  is an automorphism of  $L$  or  $\tau$  is an anti-automorphism of  $L$ .*

**Conjecture 3.2.** *For every automorphic Moufang loop all half-isomorphisms are trivial.*

In order to prove this Theorem we need to collect some results.

We will use the notation  $[u, v, w] = [[u, v]w]$  and by induction

$$[a_1, \dots, a_{n+1}] = [[a_1, \dots, a_n], a_{n+1}]$$

for left-normed commutators.

**Lemma 3.3.** (Bruck) *Let  $L$  be a left automorphic Moufang loop and assume  $t, u, v \in L$ . Then the following statements hold:*

- (i) *If  $L$  is generated by 3 elements, then the associator subloop  $(L, L, L)$  lies in the center of  $L$ .*
- (ii)  $[u, v] \in N(L)$ .
- (iii)  $[uv, t] = [u, t][u, t, v][v, t]$ ,
- (iv)  $(nu, v, t) = (u, nv, t) = (u, v, nt) = (u, v, t)$  for all  $n \in N(L)$ .
- (v) *If  $L$  is an automorphic Moufang loop, then  $u^3 \in N(L)$ .*

**Proof.** [1], chapter VII, Lemma 2.2

**q.e.d.**

Quite recently in the theory of finite Moufang loops a breakthrough has been reached: the Theorem of Lagrange and an important part of the Theorems of Sylow have been proved (see [9], [4], [10], [3]). In the following Lemma 3.4 and Lemma 3.5 these facts are used.

**Lemma 3.4.** *Let  $L$  be a finite automorphic Moufang loop, let  $\tau$  be a half-automorphism of  $L$  and let  $S$  be a 3-Sylow subloop of  $L$ . Then*

- (i)  $L = SN(L) = N(L)S$ ,
- (ii)  $\tau(L, L, L) = (\tau(L), \tau(L), \tau(L))$ ,
- (iii)  $\tau(N(L)) = N(\tau(L))$ ,
- (iv)  $\tau(C(L)) = C(\tau(L))$ ,
- (v)  $\tau[L, L] = [\tau(L), \tau(L)]$ .

**Proof.** It is known by (v) from the previous Lemma 3.3 that any finite automorphic Moufang loop  $L$  is the product of  $S$  and the nucleus  $N(L)$  of  $L$ . Thus (i) holds.

Obviously  $\tau(S)$  is a 3-Sylow subloop of  $\tau(L)$ . Applying (i) to  $\tau(L)$  we obtain

$$\tau(L) = \tau(S)\tau(N(L)) = \tau(S)N(\tau(L)).$$

For arbitrary elements  $u, v, w \in L$  we put  $u = s_1n_1$ ,  $v = s_2n_2$ ,  $w = s_3n_3$  with  $s_i \in S$ ,  $n_i \in N(L)$ ,  $1 \leq i \leq 3$ . Hence  $(u, v, w) = (s_1, s_2, s_3)$  according to Lemma 3.3,(iv). In [5] it was shown that for Moufang loop of odd order any half-automorphism is trivial, so  $\tau$  is trivial on  $S$ . If  $\tau$  is an isomorphism then

$$\tau(u, v, w) = \tau(s_1, s_2, s_3) = (\tau(s_1), \tau(s_2), \tau(s_3)) = (\tau(u), \tau(v), \tau(w)).$$

Otherwise straightforward computation shows that

$$\tau(u, v, w) = (\tau(u)^{-1}, \tau(v)^{-1}, \tau(w)^{-1})^{-1}.$$

Hence in both cases we have

$$\tau(L, L, L) = (\tau(L), \tau(L), \tau(L)) \text{ and } \tau(N(L)) = N(\tau(L))$$

by the definition of  $N(L)$ .

It is evident that  $\tau(Z(L)) = Z(\tau(L))$  and  $\tau(C(L)) = C(\tau(L))$ .

Finally,  $\tau[L, L] = [\tau L, \tau L]$  since Moufang loops are diassociative. **q.e.d.**

In the Theory of Loops there exist different notions of nilpotency ([1],[14] and [13]) which for groups, but not for loops in general, are equivalent. Here we will use the following definition:

A Moufang loop  $M$  is *commutatively nilpotent* if for some  $n \in \mathbb{N}$  any left-normed commutator word of length  $n$  is trivial on  $M$ .

**Lemma 3.5.** *Let  $L$  be a finite commutatively nilpotent automorphic Moufang loop. Then*

- (i)  *$L$  is the direct product of its 3-Sylow subloop and its 3'-Hall subgroup, which is contained in the nucleus of  $L$ .*
- (ii)  *$L$  is the direct product of its Sylow subloops.*

**Proof.** First we show (i). By Lemma 3.4 for every finite automorphic Moufang loop one has  $L = SN(L)$ , where  $S$  is a 3-Sylow subloop of  $L$ . Let  $a \in S$  and  $b \in L$  such that the orders of  $a$  and  $b$  are coprime. The elements  $a$  and  $b$  generate a nilpotent subgroup of  $L$ . Thus they commute. On the other hand,  $N(L)$  is a nilpotent subgroup of  $L$ . Hence  $N(L)$  is the direct product of its Sylow subgroups. Let  $\hat{N}$  be the direct product of all Sylow subgroups of  $N(L)$  apart of its 3-Sylow subgroup. Then  $L = S\hat{N}$  and  $S \cap \hat{N} = 1$ . Since  $\hat{N}$  is subgroup of  $N(L)$  it follows immediately that every element  $a \in L$  has a unique presentation in the product  $S\hat{N}$ . Indeed  $s_1n_1 = s_2n_2$  for  $n_1, n_2 \in \hat{N}$ , and  $s_1, s_2 \in S$  implies  $s_2^{-1}s_1 = n_2n_1^{-1}$ .

Using again the fact that  $\hat{N} \leq N(L)$  by straightforward verification one can show that  $S$  is invariant under the inner mappings group  $\text{Inn}(L)$  and  $\hat{N}$  is also invariant under the inner mappings group  $\text{Inn}(L)$ . For example,

$$l_{a,b}(s) = (ab)^{-1}(a(b(s))) = (s_1n_1 \cdot s_2n_2)^{-1}(s_1n_1(s_2n_2(s))) = s_3,$$

where  $s, s_1, s_2, s_3 \in S, n_1, n_2 \in \hat{N}$ .

It follows that  $L$  is the direct product of its 3-Sylow subloop and its 3'-Hall subgroup, which is contained in the nucleus of  $L$ . By analogous arguments one can show that every Sylow subloop is normal in  $L$ , and thus we have the second statement of the Lemma (see [1], p.72). **q.e.d.** By Lemma 3.4,(ii) one can define the induced half-isomorphism

$$\bar{\tau} : L/(L, L, L) \rightarrow \tau(L)/(\tau(L), \tau(L), \tau(L))$$

on a finite automorphic Moufang loop  $L$  with a half-isomorphism  $\tau$ . Denote  $\bar{L} = L/(L, L, L)$  and  $\tau(\bar{L}) = \tau(L)/(\tau(L), \tau(L), \tau(L))$

**Lemma 3.6.** *Let  $L$  be a finite automorphic Moufang loop with a non-trivial half-isomorphism  $\tau$ . Consider the induced half-isomorphism*

$$\bar{\tau} : \bar{L} \rightarrow \tau(\bar{L}).$$

*Then the following statements hold:*

(i)  $\bar{\tau}$  is a trivial half-isomorphism.

(ii) *If  $L$  is generated by 3 elements and if  $\bar{\tau}$  is an isomorphism on the group  $\bar{L}$ , then for every pair  $u, v \in L$  satisfying the condition*

$$\tau(uv) = \tau(v)\tau(u)$$

*one has  $[u, v] \in C(L)$  and  $[\tau(u), \tau(v)] \in C(\tau(L))$ .*

(iii) *If  $L$  is generated by 3 elements and if  $\bar{\tau}$  is an anti-isomorphism, then  $[w, t] \in C(L)$  and  $[\tau(w), \tau(t)] \in C(\tau(L))$  for all  $w, t \in L$  satisfying the condition*

$$\tau(wt) = \tau(w)\tau(t)$$

**Proof.** (i) Since  $L/(L, L, L)$  is a group it follows from Scott's Theorem [18] that  $\bar{\tau}$  is an isomorphism or an anti-isomorphism.

(ii) Assume first that  $\bar{\tau}$  is an isomorphism. For  $u, v \in L$  and  $\bar{u} = u(L, L, L)$ ,

$\bar{v} = v(L, L, L)$  we get  $\bar{\tau}(\bar{u}\bar{v}) = \bar{\tau}(\bar{u})\bar{\tau}(\bar{v})$ . We have on the other hand  $\bar{\tau}(\bar{u}\bar{v}) = \bar{\tau}(\bar{v})\bar{\tau}(\bar{u})$ . Thus the commutator  $[\bar{\tau}(\bar{u}), \bar{\tau}(\bar{v})]$  is trivial, and  $[\tau(u), \tau(v)]$  is in the  $(\tau(L), \tau(L), \tau(L)) = \tau(L, L, L)$  and consequently by Lemma 3.3, (i) one knows that  $[\tau(u), \tau(v)]$  is in the center of  $\tau(L)$ . Using half-isomorphism  $(\tau)^{-1}$  analogously one can show that  $[u, v] \in C(L)$ . Hence statement (ii) is shown.

As we mentioned above every anti-isomorphism is the composition of some isomorphism and inverse mapping, so one can treat the case (iii) of an anti-isomorphism  $\bar{\tau}$  in a similar way. **q.e.d.**

In what follows we will consider the case that  $\bar{\tau}$  is an isomorphism on  $\bar{L}$ . If  $\bar{\tau}$  is an anti-isomorphism on  $\bar{L}$  one can prove the Theorem 3.1 using the inverse mapping and analogous arguments.

**Lemma 3.7.** *Let  $L$  be a finite automorphic Moufang loop generated by 3 elements with a nontrivial half-isomorphism  $\tau$ . Put*

$$D(L, \tau) = \{g \in L \mid \exists h \in L : \tau(gh) = \tau(h)\tau(g) \neq \tau(g)\tau(h)\}.$$

*If  $d \in [L, L]$ , then  $[d, D(L, \tau)] \subseteq C(L)$ .*

**Proof.** Let  $d \in [L, L]$ ,  $g \in D(L, \tau)$ , choose  $h \in L$  such that  $\tau(gh) = \tau(h)\tau(g) \neq \tau(g)\tau(h)$ . Since  $[L, L] \subseteq N(L)$  a subloop  $L_0$  generated by  $d, h, g$  is a group. By Scott's Theorem  $\tau$  restricted to  $L_0$  is an anti-monomorphism. Hence  $\tau(dg) = \tau(g)\tau(d)$ . If  $\tau(g)\tau(d) \neq \tau(d)\tau(g)$ , then by Lemma 3.6 we get  $[g, d] \in C(L)$ . Finally, if  $\tau(g)\tau(d) = \tau(d)\tau(g)$ , then  $[g, d] = 1$ . **q.e.d.**

**Lemma 3.8.** (Gagola-Giuliani [5]) *Let  $M$  be a finite Moufang loop. Let  $\tau$  be a half-isomorphism of  $M$ . Then*

- (i)  *$\tau$  is a semi-isomorphism of  $M$ , i.e. for any  $u, v \in M$   $\tau(uvu) = \tau(u)\tau(v)\tau(v)$*
- (ii) *If  $M$  is of odd order, then  $\tau$  is trivial. In this case in particular  $\tau(M)$  is isomorphic to  $M$ .*
- (iii) *If  $\tau$  is not trivial, then there exist  $x, y, z \in M$  such that  $[x, y] \neq 1$ ,  $[x, z] \neq 1$  and*

- (1)  $\tau(xy) = \tau(x)\tau(y) \neq \tau(y)\tau(x),$
- (2)  $\tau(xz) = \tau(z)\tau(x) \neq \tau(x)\tau(z).$

**q.e.d.**

Let us call such elements  $\{x, y, z\}$  as it is described above in the Lemma 3.8 a *Gagola-Giuliani triple* or *GG-triple* for short.

**Lemma 3.9.** *Let  $L$  be a finite automorphic Moufang loop with a non-trivial half-automorphism  $\tau$ . A subloop of  $L$  generated by a Gagola-Giuliani triple is commutatively nilpotent.*

**Proof.** Let  $L$  be a finite automorphic Moufang loop with non-trivial half-automorphism  $\tau$  and let  $\{x, y, z\}$  be a GG-triple of a loop  $L$ . Let us recall that this means that conditions (iii) of Lemma 3.8 hold. Hence (1) implies that  $\tau|_{\langle x, y \rangle}$  is a monomorphism and (2) implies that  $\tau|_{\langle x, z \rangle}$  is an anti-monomorphism. Thus there are two possibilities for the subgroup  $\langle y, z \rangle$ : either  $\tau|_{\langle y, z \rangle}$  is a monomorphism or  $\tau|_{\langle y, z \rangle}$  is an anti-monomorphism.

Put  $\tau(x) = a$ ,  $\tau(y) = b$ ,  $\tau(z) = c$ . Note that  $\{a, b, c\}$  is a GG-triple if and only if  $\{x, y, z\}$  is a GG-triple. Indeed the conditions (1) and (2) of Lemma 3.8 are equivalent to

$$\begin{aligned} \tau^{-1}(ab) &= \tau^{-1}(a)\tau^{-1}(b) \neq \tau^{-1}(b)\tau^{-1}(a) \\ \tau^{-1}(ac) &= \tau^{-1}(c)\tau^{-1}(a) \neq \tau^{-1}(a)\tau^{-1}(c) \end{aligned}$$

Let  $L_1$  be a loop generated  $x, y, z$  and  $L_2$  be a loop generated by  $a, b, c$ . Obviously the mappings  $\tau|_{L_1} : L_1 \rightarrow L_2$  and  $\tau^{-1}|_{L_2} : L_2 \rightarrow L_1$  are non-trivial half-isomorphisms.

**Case 1:** Suppose that  $\tau|_{\langle y, z \rangle}$  is an anti-monomorphism and

$$\tau(yz) = \tau(z)\tau(y) = cb \neq \tau(y)\tau(z) = bc.$$

One has  $[L_2, L_2, u] \subseteq C(L_2)$ , where  $u \in \{a, b, c\} \subseteq D(L_2, \tau^{-1})$  by Lemma 3.7. Hence  $[L_2, L_2, L_2] \subseteq C(L_2)$  by Lemma 3.3,(iii) and  $L_2$  is commutatively nilpotent of class  $\leq 4$ .

**Case 2:** Suppose that  $\tau|_{\langle y, z \rangle}$  is a monomorphism. Let us study the behavior of the half-isomorphism  $\tau$  restricted to the following two-generated subgroup:

$$L_3 = \langle xy, zx \rangle,$$

Suppose that  $\tau$  restricted to  $L_3$  is a monomorphism. Then

$$\tau(xy \cdot zx) = \tau(xy)\tau(zx) = ab \cdot ac.$$

By the Moufang identity and by Lemma 3.8(i),

$$\tau(xy \cdot zx) = \tau(x \cdot yz \cdot x) = a \cdot bc \cdot a = ab \cdot ca.$$

Hence  $ac = ca$  which contradicts the condition that  $\{x, y, z\}$  is GG-triple. Therefore  $\tau$  restricted to  $L_3$  is an anti-monomorphism. Now by Lemma 3.6 one has  $[ab, ac] \in C(L_2)$  as well as  $[a, c] \in C(L_2)$ . But it can also be proven that

$$[ab, ac] = [a, c] \neq 1.$$

In order to see this, note that

$$\begin{aligned} ac \cdot ab &= \tau(zx)\tau(xy) = \tau(xy \cdot zx) \\ &= \tau(x \cdot yz \cdot x) = a \cdot bc \cdot a = ab \cdot ca, \end{aligned}$$

Let us consider the GG-triple  $\{a, ab, ac\}$ . The map  $\tau^{-1}$  restricted to  $\langle a, ab \rangle$  is a monomorphism,  $\tau^{-1}$  restricted to  $\langle a, ac \rangle$  is an anti-monomorphism, and  $\tau^{-1}$  restricted to  $\langle ab, ac \rangle$  is an anti-monomorphism too. Thus we have the conditions of the Case 1 and therefore the loop generated by  $\{a, ab, ac\}$  is commutatively nilpotent. But

$$L_2 = \langle a, ab, ac \rangle = \langle a, b, c \rangle.$$

The Lemma is proved. **q.e.d.**

Now everything is ready to prove the Theorem 3.1.

By Lemma 3.5 the loop  $L_2$  is a direct product of its Sylow subloops. So  $\tau^{-1}$  acts componentwise on  $L_2$ . All Sylow subloops except the 3-Sylow subloop are groups, and the 3-Sylow subloop is a loop of odd



order. By the Lemma 3.8(ii)  $\tau^{-1}$  is a trivial half-isomorphism on  $L_2$ , which forms a contradiction with the conditions of Lemma 3.9. Thus Theorem 3.1 is proved. **q.e.d.**

#### 4. EXAMPLES.

We give two examples of loops which admit a non-trivial half-automorphism. Both examples are in a different way not too far away from being automorphic Moufang loops. All our statements about these examples are easily checked using the LOOPS package of GAP [15]

1. Let  $Q_1$  be the loop of order 16 given by the Cayley table

| $\circ$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1       | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 2       | 2  | 4  | 8  | 6  | 3  | 1  | 5  | 7  | 14 | 9  | 16 | 10 | 11 | 12 | 13 | 15 |
| 3       | 3  | 5  | 4  | 7  | 6  | 8  | 1  | 2  | 15 | 13 | 9  | 11 | 14 | 16 | 12 | 10 |
| 4       | 4  | 6  | 7  | 1  | 8  | 2  | 3  | 5  | 12 | 14 | 15 | 9  | 16 | 10 | 11 | 13 |
| 5       | 5  | 7  | 2  | 8  | 4  | 3  | 6  | 1  | 13 | 11 | 14 | 16 | 12 | 15 | 10 | 9  |
| 6       | 6  | 1  | 5  | 2  | 7  | 4  | 8  | 3  | 10 | 12 | 13 | 14 | 15 | 9  | 16 | 11 |
| 7       | 7  | 8  | 1  | 3  | 2  | 5  | 4  | 6  | 11 | 16 | 12 | 15 | 10 | 13 | 9  | 14 |
| 8       | 8  | 3  | 6  | 5  | 1  | 7  | 2  | 4  | 16 | 15 | 10 | 13 | 9  | 11 | 14 | 12 |
| 9       | 9  | 10 | 11 | 12 | 16 | 14 | 15 | 13 | 4  | 6  | 7  | 1  | 5  | 2  | 3  | 8  |
| 10      | 10 | 12 | 16 | 14 | 15 | 9  | 13 | 11 | 2  | 4  | 5  | 6  | 3  | 1  | 8  | 7  |
| 11      | 11 | 13 | 12 | 15 | 10 | 16 | 9  | 14 | 3  | 8  | 4  | 7  | 6  | 5  | 1  | 2  |
| 12      | 12 | 14 | 15 | 9  | 13 | 10 | 11 | 16 | 1  | 2  | 3  | 4  | 8  | 6  | 7  | 5  |
| 13      | 13 | 15 | 10 | 16 | 9  | 11 | 14 | 12 | 8  | 7  | 2  | 5  | 4  | 3  | 6  | 1  |
| 14      | 14 | 9  | 13 | 10 | 11 | 12 | 16 | 15 | 6  | 1  | 8  | 2  | 7  | 4  | 5  | 3  |
| 15      | 15 | 16 | 9  | 11 | 14 | 13 | 12 | 10 | 7  | 5  | 1  | 3  | 2  | 8  | 4  | 6  |
| 16      | 16 | 11 | 14 | 13 | 12 | 15 | 10 | 9  | 5  | 3  | 6  | 8  | 1  | 7  | 2  | 4  |

Then  $Q_1$  is a Code loop (see [7]) in particular it is a left automorphic Moufang loop. One can see that the mapping  $\phi : Q_1 \rightarrow Q_1$  defined by

$$\phi(5) = 8, \phi(8) = 5, \phi(x) = x \text{ for } x \notin \{5, 8\}$$

is a half-automorphism. Since

$$\phi(2 \circ 7) = \phi(5) = 8 = \phi(7) \circ \phi(2) \neq \phi(2) \circ \phi(7) = 5$$

$$\phi(3 \circ 9) = \phi(15) = 15 = \phi(3) \circ \phi(9) \neq \phi(9) \circ \phi(3) = 11$$

this half-automorphism is non-trivial.

2. The loop  $Q_2$  defined by the Cayley table

| $\circ$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---------|---|---|---|---|---|---|---|---|
| 1       | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2       | 2 | 1 | 4 | 3 | 6 | 5 | 8 | 7 |
| 3       | 3 | 4 | 1 | 2 | 7 | 8 | 6 | 5 |
| 4       | 4 | 3 | 2 | 1 | 8 | 7 | 5 | 6 |
| 5       | 5 | 6 | 8 | 7 | 1 | 2 | 4 | 3 |
| 6       | 6 | 5 | 7 | 8 | 2 | 1 | 3 | 4 |
| 7       | 7 | 8 | 5 | 6 | 3 | 4 | 2 | 1 |
| 8       | 8 | 7 | 6 | 5 | 4 | 3 | 1 | 2 |

is an automorphic loop, but not a Moufang loop and the permutation  $\phi = (3, 5)(4, 6)(7, 8)$  is a half-automorphism of  $Q_2$ . By

$$\phi(4 \circ 6) = \phi(7) = 8 = \phi(4) \circ \phi(6) \neq \phi(6) \circ \phi(4) = 7$$

$$\phi(4 \circ 8) = \phi(6) = 4 = \phi(8) \circ \phi(4) \neq \phi(4) \circ \phi(8) = 3$$

$\phi$  is no-trivial half-automorphism.

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